# Refresher on the Syntax, Semantics, and Derivation Rules of Propositional and Predicate Logic

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## 1. Syntax of PL

The syntactic rules for $PL$ (predicate logic) defines the class of well-formed propositions (PL sentences, formulas, etc.)[1] They say what counts as the basic expressions of PL, and the rules for generating more complex expressions.

### A. 'Lexicon' of $PL$

(1)

| *Category* | *Type* | *Basic expressions* |
|---|---|---|
| Names/individual constants | Term | $d, n, j, m$ |
| (Individual) variables | Term | $x, y, z, ...$ |
| One-place predicate | Pred | $M, B$ |
| Two-place predicate | Pred | $K, L$ |

### B. Formation rules of $PL$

(2)
- i. If $\delta$ is a one-place predicate and $\alpha$ is a term, then $\delta(\alpha)$ is a sentence.
- ii. If $\gamma$ is a two-place predicate and $\alpha$ and $\beta$ are terms, then $\gamma(\alpha, \beta)$ is a sentence.
- iii. If $\phi$ is a sentence, then $\neg\phi$ is a sentence.
- iv. If $\phi$ and $\psi$ are sentences, then $(\phi \wedge \psi)$ is a sentence.
- v. If $\phi$ and $\psi$ are sentences, then $(\phi \vee \psi)$ is a sentence.
- vi. If $\phi$ and $\psi$ are sentences, then $(\phi \rightarrow \psi)$ is a sentence.
- vii. If $\phi$ and $\psi$ are sentences, then $(\phi \leftrightarrow \psi)$ is a sentence.
- viii. If $v$ is a variable, and $\phi$ is a sentence, then $\forall v\phi$ is a sentence.
- ix. If $v$ is a variable, and $\phi$ is a sentence, then $\exists v\phi$ is a sentence.
- x. Nothing else is a sentence.

[1] I follow Dowty, Wall, and Peters' (**?**) formulation, with modifications and additions based on a handout by Barbara Partee (`http://people.umass.edu/partee/RGGU_2008/RGGU081.pdf`).

**Some notes**:

- Expanding the language to arbitrary $n$-place predicates can be accomplished by generalizing (2i)-(2ii) as in:
  - If $\rho$ is an $n$-place predicate, and $\alpha_1, ..., \alpha_n$ are terms, then $\rho(\alpha_1, ...\alpha_n)$ is a sentence.

- Parentheses are omitted in many uses of PL:

  e.g. where in this handout I have written things like $K(d, j)$, others might write it $Kdj$ or even $dKj$.

Some sentences of $PL$, with some of "the proof" that they are well-formed given

(3) a. $M(d)$ — (2i)
b. $K(d, j)$ — (2ii)
c. $\neg M(d)$ — (3a), (2iii)
d. $\neg K(d, j)$ — (3b), (2iii)
e. $\forall x(L(d, x) \rightarrow B(x))$ — (2i), (2ii), (2vi), (2viii)
f. $\exists y(B(y) \wedge K(m, y))$ — (2i), (2ii), (2iv), (2ix)

## 2. Semantics of PL

The semantic component determines an interpretation relative to a **model**, $\mathcal{M}$:

$$\mathcal{M} = \langle D, I \rangle$$

- $D$ is the **set of entities** we can use the language to talk about
- $I$ is a **lexical interpretation function** that assigns semantic values to the constants
- There is a set $G$ of **assignment functions**, $g$, that map variables to entities in $D$.
  - e.g., $g_9 = \{\langle x, d \rangle, \langle y, n \rangle, \langle z, j \rangle, \langle w, m \rangle\}$
  - e.g., $g_{946} = \{\langle x, n \rangle, \langle y, j \rangle, \langle z, m \rangle, \langle w, d \rangle\}$, etc.
- A **semantic interpretation**, $[\![\cdot]\!]^{\mathcal{M},g}$, is built up recursively on the basis of $I$, and assigns to every expression $\alpha$ its semantic value in $\mathcal{M}$ relative to some $g$, i.e.:

$$\text{The interpretation of } \alpha \text{ is } [\![\alpha]\!]^{\mathcal{M},g}.$$

A. **'Lexical' rules** of $PL$

(4) a. If $\alpha$ is a constant, then $[\![\alpha]\!]^{\mathcal{M},g} = I(\alpha)$.
b. If $\alpha$ is a variable, then $[\![\alpha]\!]^{\mathcal{M},g} = g(\alpha)$.

B. **Composition rules of** $PL$

(5) i. If $\delta$ is a one-place predicate and $\alpha$ is a name,
then $[\![\delta(\alpha)]\!]^{\mathcal{M},g} = \top$ iff $[\![\alpha]\!]^{\mathcal{M},g} \in [\![\delta]\!]^{\mathcal{M},g}$.

ii. If $\gamma$ is a two-place predicate and $\alpha$ and $\beta$ are names,
then $[\![\gamma(\alpha,\beta)]\!]^{\mathcal{M},g} = \top$ iff $\langle [\![\alpha]\!]^{\mathcal{M},g}, [\![\beta]\!]^{\mathcal{M},g} \rangle \in [\![\gamma]\!]^{\mathcal{M},g}$.

iii. If $\phi$ is a sentence,
then $[\![\neg\phi]\!]^{\mathcal{M},g} = \top$ iff $[\![\phi]\!]^{\mathcal{M},g} =\perp$.

iv. If $\phi$ and $\psi$ are sentences,
then $[\![(\phi \wedge \psi)]\!]^{\mathcal{M},g} = \top$ iff $[\![\phi]\!]^{\mathcal{M},g} = \top$ and $[\![\psi]\!]^{\mathcal{M},g} = \top$.

v. If $\phi$ and $\psi$ are sentences,
then $[\![(\phi \vee \psi)]\!]^{\mathcal{M},g} = \top$ iff either $[\![\phi]\!]^{\mathcal{M},g} = \top$ or $[\![\psi]\!]^{\mathcal{M},g} = \top$.

vi. If $\phi$ and $\psi$ are sentences,
then $[\![(\phi \rightarrow \psi)]\!]^{\mathcal{M},g} = \top$ iff either $[\![\phi]\!]^{\mathcal{M},g} =\perp$ or $[\![\psi]\!]^{\mathcal{M},g} = \top$.

vii. If $\phi$ and $\psi$ are sentences,
then $[\![(\phi \leftrightarrow \psi)]\!]^{\mathcal{M},g} = \top$ iff
**either** $[\![\phi]\!]^{\mathcal{M},g} = \top$ and $[\![\psi]\!]^{\mathcal{M},g} = \top$ **or** $[\![\phi]\!]^{\mathcal{M},g} =\perp$ and $[\![\psi]\!]^{\mathcal{M},g} =\perp$.

viii. If $\alpha$ is a variable and $\phi$ is a sentence,
then $[\![\forall\alpha\phi]\!]^{\mathcal{M},g} = \top$ iff for all $e \in D$, $[\![\phi]\!]^{\mathcal{M},g[e/\alpha]} = \top$.

ix. If $\alpha$ is a variable and $\phi$ is a sentence,
then $[\![\exists\alpha\phi]\!]^{\mathcal{M},g} = \top$ iff there is a $e \in D$, $[\![\phi]\!]^{\mathcal{M},g[e/\alpha]} = \top$.

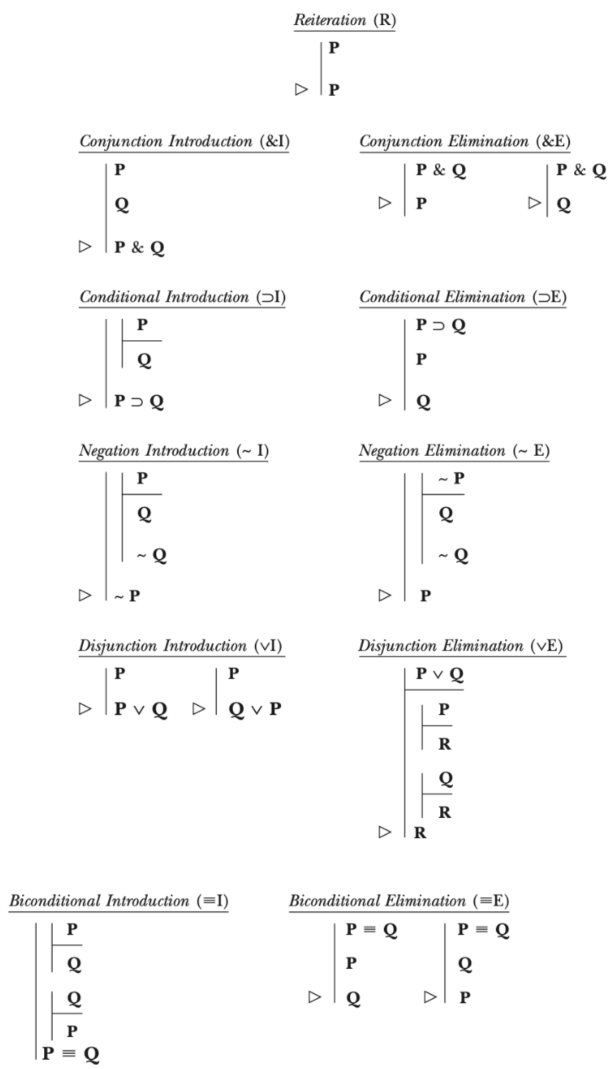
- **Some notes**:
  - The Greek lowercase variables are used in the metalanguage as variables over *expressions*. Otherwise, regular math mode lowercase letters are used (e.g. the $e$s in (5viii) and (5ix) range over individuals in $D$, not expressions).
  - If the language is expanded to allow arbitrary $n$-place predicates, (5i)-(5iii) generalize as:
    * If $\rho$ is an $n$-place predicate and $\alpha_1, ..., \alpha_n$ are names,
      then $[\![\rho(\alpha_1, ...\alpha_n)]\!]^{\mathcal{M},g} = 1$ iff $\langle [\![\alpha_1]\!]^{\mathcal{M},g}, ..., [\![\alpha_n]\!]^{\mathcal{M},g} \rangle \in [\![\rho]\!]^{\mathcal{M},g}$
  - All of rules (5iii)-(5ix) are **syncategorematic** interpretations—this type of interpretation is only **necessary** (it seems) for (5viii) and (5ix), though.

- Rules (5iii)-(5vii) have the effect of the familiar truth tables for the connectives from Propositional Logic.

- **The notation** $g[e/\alpha]$ **means**: 'the variable assignment that is just like $g$ except for the (possible) difference that $g[e/\alpha]$ assigns the individual $e$ to $\alpha$'.
  - For any name $\alpha$, $[\![\alpha]\!]^{\mathcal{M},g[e/x]} = I(\alpha)$.
  - For any variable $\alpha$, $[\![\alpha]\!]^{\mathcal{M},g[e/\alpha]} = g[e/\alpha](\alpha) = e$.

## 3. Proof rules cheatsheet

**Derivation rules of propositional logic**



## Derivation rules of predicate logic

*Substitution instance of* **P**: If **P** is a sentence of *PLE* of the form $(\forall \mathbf{x})\mathbf{Q}$ or $(\exists \mathbf{x})\mathbf{Q}$ and **t** is a closed individual term, then $\mathbf{Q}(\mathbf{t}/\mathbf{x})$ is a substitution instance of **P**. The individual term **t** is the *instantiating individual term.*

*Universal Elimination* ($\forall$E)

$$
\begin{array}{l|l}
 & (\forall \mathbf{x})\mathbf{P} \\
\triangleright & \mathbf{P}(\mathbf{a}/\mathbf{x})
\end{array}
$$

*Existential Introduction* ($\exists$I)

$$
\begin{array}{l|l}
 & \mathbf{P}(\mathbf{a}/\mathbf{x}) \\
\triangleright & (\exists \mathbf{x})\mathbf{P}
\end{array}
$$

*Universal Introduction* ($\forall$I)

$$
\begin{array}{l|l}
 & \mathbf{P}(\mathbf{a}/\mathbf{x}) \\
\triangleright & (\forall \mathbf{x})\mathbf{P}
\end{array}
$$

provided that

(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\forall \mathbf{x})\mathbf{P}$.

*Existential Elimination* ($\exists$E)

$$
\begin{array}{l|l}
 & (\exists \mathbf{x})\mathbf{P} \\
 & \begin{array}{|l} \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \hline \mathbf{Q} \end{array} \\
\triangleright & \mathbf{Q}
\end{array}
$$

provided that

(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\exists \mathbf{x})\mathbf{P}$.
(iii) **a** does not occur in **Q**.